

Theoretical Analysis of Subthreshold Oscillatory Behaviors in Nonlinear Autonomous Systems

Shenbing Kuang, Jiafu Wang,* Ting Zeng, and Aiyin Cao

Department of Physical Science and Technology, Wuhan University of Technology, Wuhan 430070 China

We have developed a linearization method to investigate the subthreshold oscillatory behaviors in nonlinear autonomous systems. By considering firstly the neuronal system as an example, we show that this theoretical approach can predict quantitatively the subthreshold oscillatory activities, including the damping coefficients and the oscillatory frequencies which are in good agreement with those observed in experiments. Then we generalize the linearization method to an arbitrary autonomous nonlinear system. The detailed extension of this theoretical approach is also presented and further discussed.

PACS numbers: 87.10.+e, 05.45.-a

Oscillatory behaviors are one of the most important features for the nonlinear systems and have attracted much attention in recent years [1, 2, 3]. Oscillation phenomena have been discovered in many areas of the chemical and biological sciences, such as chemical oscillation [4, 5], rhythmic gene expression and metabolism [6], and particularly neuronal oscillations [7, 8, 9]. These oscillatory behaviors may have some observable influences on the responses of the systems, and play significant and often crucial roles, for example in neuronal system the oscillation activities are found to be important for information processing [10, 11] and cognitive perception [12, 13, 14, 15]. It is therefore of great interest to study the oscillatory behaviors in these nonlinear systems. Oscillation theory has become an important part of contemporary applied mathematics [16, 17] and has been well developed [18, 19]. However, the analysis of oscillatory processes is still far from being complete. Up to date, little attention has been paid to the question how to systematically characterize the oscillatory behaviors, especially regarding the damping coefficients and oscillatory frequencies, in these nonlinear systems.

Generally, the dynamics of nonlinear systems are often described by some coupled differential equations [18, 19]. While the stimuli to the system involve no variables that are differentiating (constant stimuli for example) we call the nonlinear system an autonomous one. In this Letter we attempt to approach the aforementioned issue by presenting a linearization method to theoretically investigate the subthreshold oscillation behaviors in nonlinear autonomous systems. Firstly we consider the oscillatory activities in the neuronal system under subthreshold constant stimulus as an example. The theoretical approach enable us to obtain analytical solutions for the damping coefficients and frequencies of oscillatory behaviors of membrane voltage. In addition, we also generalize the theoretical approach to an arbitrary nonlinear autonomous system. The detailed implementation of this theoretical framework is also presented and discussed.

Neuronal systems are highly nonlinear excitable systems. To elucidate the subthreshold oscillatory behav-

iors in autonomous neuronal system, we consider the Hindmarsh-Rose (HR) neural model, as an example, under a subthreshold constant stimulus. The dynamics of HR neuron system could be described by the following equations[20]:

$$\begin{aligned} dx/dt &= y - z - ax^3 + bx^2 + I(t), \\ dy/dt &= c - dx^2 - y, \\ dz/dt &= r[s(x - x_0) - z], \end{aligned} \quad (1)$$

where x denotes the membrane potential, y the recovery variable and z a slow adaptation variable. and the parameters a , b , c , d , r , s and x_0 are the same as in Ref.[20]. If the membrane potential $x(t)$ exceeds its threshold value $x_{th} = 0$, the neuron will result in a spike $S(t) = \theta(x(t) - x_{th})$, where $\theta(x)$ is the Heaviside function. $I(t)$ represents the stimulus bias to the neuron, $I(t) = I_0 < I_c$ with I_c being the critical value for constant stimulus.

It is well known that the HR neuron will be active with limit-circle firings [21] for a suprathreshold constant input, $I_0 > I_c$ in Eqs. (1). If $I_0 < I_c$, the system may undergo a temporal process of damping oscillation to reach a quiescent stable state, with the membrane potential $x(t)$, the recovery variable $y(t)$ and the slow adaptation variable $z(t)$ being the stable value x_s , y_s , z_s , respectively. These stable values can be obtained from Eqs. (1) by simply letting $dx/dt = dy/dt = dz/dt = 0$, yielding

$$ax_s^3 + (d - b)x_s^2 + sx_s - (sx_0 + c) = I_0 \quad (2)$$

and $y_s = c - dx_s^2$, $z_s = s(x_s - x_0)$. It is easy to prove that Eq. (2), with respect to x_s , has only one real root for any value of I_0 ,

$$x_s = (-Q/2 + \sqrt{\Delta})^{1/3} + (-Q/2 - \sqrt{\Delta})^{1/3} - (d - b)/3a \quad (3)$$

where $\Delta = (Q/2)^2 + [s/3a - (d - b)^2/9a^2]^3$ and $Q = 2(d - b)^3/27a^3 - s(d - b)/3a^2 - (sx_0 + c + I_0)/a$. One can always write

$$\begin{aligned} x(t) &= x_s + \tilde{x}(t), \\ y(t) &= y_s + \tilde{y}(t), \\ z(t) &= z_s + \tilde{z}(t), \end{aligned} \quad (4)$$

Since $I_0 < I_{c0}$, the variables $\tilde{x}(t)$, $\tilde{y}(t)$ and $\tilde{z}(t)$ are small (approximately zero) for sufficiently large t ; so one can linearize Eqs. (1), by utilizing Eqs. (4) into the following equation

$$\begin{bmatrix} d\tilde{x}/dt \\ d\tilde{y}/dt \\ d\tilde{z}/dt \end{bmatrix} = \begin{bmatrix} 2bx_s - 3ax_s^2 & 1 & -1 \\ -2dx_s & -1 & 0 \\ rs & 0 & -r \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} \quad (5)$$

The eigenvalue λ of the matrix in Eqs. (5) yields

$$(\lambda+r)[(\lambda+1)(\lambda+3ax_s^2-2bx_s)+2dx_s]+rs(\lambda+1)=0 \quad (6)$$

When $0 \leq I_0 < I_c$, Eq. (6) has one real root, which remains negative, and two conjugated complex roots, say $\lambda_r \pm i\lambda_i$. These roots, as well as x_s , depend on I_0 and can be obtained analytically through Eqs. (3) and (6):

$$\begin{aligned} \lambda_r(I_0) &= -[(-\epsilon/2 + \sqrt{\delta})^{1/3} + (-\epsilon/2 - \sqrt{\delta})^{1/3}]/2 - \alpha/3, \\ \lambda_i(I_0) &= \sqrt{3}[(-\epsilon/2 - \sqrt{\delta})^{1/3} - (-\epsilon/2 + \sqrt{\delta})^{1/3}]/2, \end{aligned} \quad (7)$$

where $\delta = (\epsilon/2)^2 + [(-\alpha^2/3 + \beta)/3]^3$, $\epsilon = 2\alpha^3/27 - \alpha\beta/3 + \gamma$, $\alpha = 3ax_s^2 - 2bx_s + r + 1$, $\beta = 3a(1+r)x_s^2 + 2(d-b)x_s + r(s+1)$ and $\gamma = r[3ax_s^2 + 2(d-b)x_s + s]$ with x_s obtained from Eq. (3).

The analytical results for the dependence of λ_r upon I_0 are shown in figure 1. When I_0 increases from 0 to the threshold value I_c , the negative λ_r increases monotonically up to the critical value 0 corresponding to the limit-circle firings behaviors observed in simulation [21]. Be aware that there still exists damped behaviors even when the input to the neuron is a inhibitory one. The imaginary part $\lambda_i \neq 0$ means that the membrane potential $x(t)$, as a function of the real time t , behaves a damping oscillation with an intrinsic frequency $f_i \equiv \lambda_i$. The calculation results of f_i versus I_0 are shown in figure 2. One can see that the intrinsic frequency f_i of the damping oscillation is about $10 \sim 40$ Hz, depending on I_0 . It is notable that the damping oscillation does exist even when $I_0 = 0$, indicating the inherence of the oscillation. Obviously, this frequency range is the same as the subthreshold activity observed in experiments [7, 22, 23, 24, 25].

In fact, our theoretical analysis of subthreshold oscillation activities in neuronal systems are naturally independent of the specific neuronal model used. The intrinsic oscillatory behaviors in other excitable neuronal models, such as the Bonhoeffer van der Pol (BvP) model, FitzHugh-Nagumo (FHN) model and Hodgkin-Huxley - type (HH) one, can also be obtained analytically or numerically provided that the stimuli to these nonlinear systems are subthreshold constant ones. In the following we will describe a generic version of our theoretical approach, for the case of an arbitrary nonlinear autonomous system under subthreshold stimuli, to investigate the intrinsic oscillatory behaviors with an emphasis on the damping coefficients and oscillatory frequencies.

The dynamics of nonlinear autonomous systems are generally described by the differential equations of the

first order or higher order for complex systems, and the latter can easily be reduced to several coupling differential equations of first order. In this study, we suppose that an autonomous nonlinear system is governed by the following n coupling equations:

$$dx_i/dt = f_i(x_1, x_2, \dots, x_n) + A_i \quad (i = 1, 2, \dots, n) \quad (8)$$

where A_i are the constant stimuli to the system and make the nonlinear system an autonomous one. We simply let $dx_i/dt = 0$ and easily get the following n equations with real coefficients:

$$f_i(x_1, x_2, \dots, x_n) = 0, \quad (i = 1, 2, \dots, n) \quad (9)$$

In principle, there are up to n real roots for the above equations, indicating that the nonlinear autonomous system has n steady states at most. Generally, the Eqs. (9) may have both real roots and complex roots, the former of which implies the number of steady states while the latter should appear in pairs of conjugated complex due to the real coefficients of Eqs. (9). If all the roots of the Eqs. (9) are complex ones, we can conclude that the dynamic of the nonlinear autonomous system will behavior divergently or chaotically and the nonlinear system is no longer convergent to any steady states, which is beyond our consideration in this study.

For simplicity let us pay close attention to one of the steady state of system, x_s , where each variables of the system reaches their own steady value, x_{is} . During the convergent process of the system one can always write

$$x_i = x_{is} + \tilde{x}_i, \quad (i = 1, 2, \dots, n) \quad (10)$$

whereas \tilde{x}_i are small enough for sufficiently long time t . Inserting Eq. (10) into Eqs. (8) we can get

$$d\tilde{x}_i/dt = f_i(x_{1s} + \tilde{x}_1, x_{2s} + \tilde{x}_2, \dots, x_{ns} + \tilde{x}_n) \quad (i = 1, 2, \dots, n) \quad (11)$$

where the expression terms on the right side of Eqs. (11) can be further expanded at the steady state, x_s , up to the accuracy of first order Taylor expansion since \tilde{x}_i are sufficient small variables, into the following as:

$$\begin{aligned} & f_i(x_{1s} + \tilde{x}_1, x_{2s} + \tilde{x}_2, \dots, x_{ns} + \tilde{x}_n) \\ &= f_i(x_{1s}, x_{2s}, \dots, x_{ns}) \\ &+ \tilde{x}_1 \frac{\partial f_i}{\partial x_1}|_{x_s} + \tilde{x}_2 \frac{\partial f_i}{\partial x_2}|_{x_s} + \dots + \tilde{x}_n \frac{\partial f_i}{\partial x_n}|_{x_s}, \quad (i = 1, 2, \dots, n) \end{aligned} \quad (12)$$

After inserting Eqs. (9) and Eqs. (12) into Eqs. (11) we obtain n new differential equations expressed below in the form of the matrix.

$$\begin{bmatrix} d\tilde{x}_1/dt \\ \dots \\ d\tilde{x}_i/dt \\ \dots \\ d\tilde{x}_n/dt \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}|_{x_s} & \dots & \frac{\partial f_1}{\partial x_i}|_{x_s} & \dots & \frac{\partial f_1}{\partial x_n}|_{x_s} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial f_i}{\partial x_1}|_{x_s} & \dots & \frac{\partial f_i}{\partial x_i}|_{x_s} & \dots & \frac{\partial f_i}{\partial x_n}|_{x_s} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1}|_{x_s} & \dots & \frac{\partial f_n}{\partial x_i}|_{x_s} & \dots & \frac{\partial f_n}{\partial x_n}|_{x_s} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \dots \\ \tilde{x}_i \\ \dots \\ \tilde{x}_n \end{bmatrix} \quad (13)$$

The eigenvalues of the $n \times n$ real matrix in Eqs. (13) can thus be expressed as a function of steady values x_s . Since the coefficients of the matrix in Eqs. (13) are real, the coefficients of the eigenfunction should also be real numbers. In general the eigenvalues of the matrix with real coefficients should be either real numbers or in pairs of conjugated complex. For the case of real numbers, the eigenvalues would remain negative, corresponding to the convergence of the nonlinear system under subthreshold stimuli. Otherwise the system may undergo divergent or chaotic dynamics; For the case of pairs of conjugated complex, the real part of eigenvalue must be negative suggesting the damping factors of the system, while the imaginary part implies the frequencies of the oscillatory activities. As a rule, there are up to $\lceil \frac{n}{2} \rceil$ components of oscillatory frequencies in this nonlinear autonomous system ($\lceil \cdot \rceil$ means the biggest integer that no greater than what's in the square bracket).

At the any other steady state beyond x_s , the same linearizaiton method could also be applied. The key of our theoretical approach is that the stimuli to the nonlinear autonomous system must be subthreshold ones, to keep the nonlinear system from divergence or chaos, so that the linearization method is always available.

In summary, we have developed a linearization method to study the subthreshold oscillatory behaviors in nonlinear autonomous systems. This theoretical approach enable us to predict the dynamics of subthreshold behaviors and quantitatively describe the damping coefficients and frequencies of oscillatory behaviors, if exist. Given the case of nonlinear neuronal system under subthreshold constant stimuli, our theoretical analysis can give analytical solutions about its oscillation behaviors that are consistent with experimental observations. Since oscillatory activities are surprisingly ubiquitous phenomena in a variety of nonlinear systems, the linearization method provide us a possible theoretical approach to characterize the oscillatory behaviors in those autonomous systems. Regarding the oscillatory behaviors in the non-autonomous nonlinear system, one have to seek other methods or solutions which are our interest of further study.

ACKNOWLEDGEMENTS

The authors would like to acknowledge the financial support from the Key Project of Chinese Ministry of Ed-

ucation (Grant No. 106115).

* jasper@whut.edu.cn

- [1] P. B. Chapman. *J. Austral. Math. Soc. Ser. B*, 35:445, 1994.
- [2] N. A. Bobylev et al. *Approximation Procedures in Nonlinear Oscillation Theory*. Walter de Gruyter, Berlin, 1994.
- [3] S. Lefschetz. *Contributions to the Theory of Nonlinear Oscillations*. Princeton University Press, Princeton, NJ, 2001.
- [4] I. Z. Kiss. *Science*, 296:1672, 2002.
- [5] H. Yui M. Fujinami Y. Ikezoe, S. Ishizaki and T. Sawada. *Analytical Sciences*, 20:1509, 2004.
- [6] U. Schibler and F. Maef. *Curr. Opin. Cell Biol.*, 17:223, 2005.
- [7] G. Buzsaki. *Neuron*, 33:325, 2002.
- [8] G. Buzsaki and A. Draguhn. *Science*, 304:1926, 2004.
- [9] E. Idou et al. *J. Neurophysiol.*, 96:175, 2006.
- [10] B. Hutcheon and Y. Yarom. *Trends Neurosci.*, 23:216, 2000.
- [11] H. Spors A. T. Schaefer, K. Angelo and T. W. Margrie. *PLoS. Biol.*, 4:1010, 2006.
- [12] L. M. Ward. *Trends Cogn. Sci.*, 7:553, 2003.
- [13] U. Ribary. *Prog. Brain Res.*, 150:127, 2005.
- [14] A. Gelperin. *J. Neurosci.*, 26:1663, 2006.
- [15] J. Kaiser and W. Lutzenberger. *Rev. Neurosci.*, 16:239, 2005.
- [16] L. Berezansky and E. Braverman. *J. Comput. Appl. Math.*, 151:119, 2003.
- [17] M. R. S. Kulenović and N. Moussa. *Appl. Math. Lett.*, 18:505, 2005.
- [18] Q. Kong L. H. Erbe and B. G. Zhang. *Oscillation Theory for Functional Differential Equations*. Marcel Dekker, New York, 1995.
- [19] I. Györi and G. Ladas. *Oscillation Theory for Delay Differential Equations*. Oxford University Press, Oxford, 1991.
- [20] Hindmarsh J. L and R. M. Rose. *Nature*, 296:162, 1982.
- [21] Hindmarsh J. L and R. M. Rose. *Proc. R. Soc. London B*, 221:87, 1984.
- [22] A. A. Grace R. R. Llinas and Y. Yarom. *Proc. Natl. Acad. Sci. (USA)*, 88:891, 1991.
- [23] I. Lampl and Y. Yarom. *J. Neurophysiol.*, 70:2181, 1993.
- [24] E. O. Mann and O. Paulsen. *Prog. Biophys. Mol. Biol.*, 87:67, 2005.
- [25] I. Vida M. Bartos and P. Jonas. *Nat. Rev. Neurosci.*, 8:45, 2007.

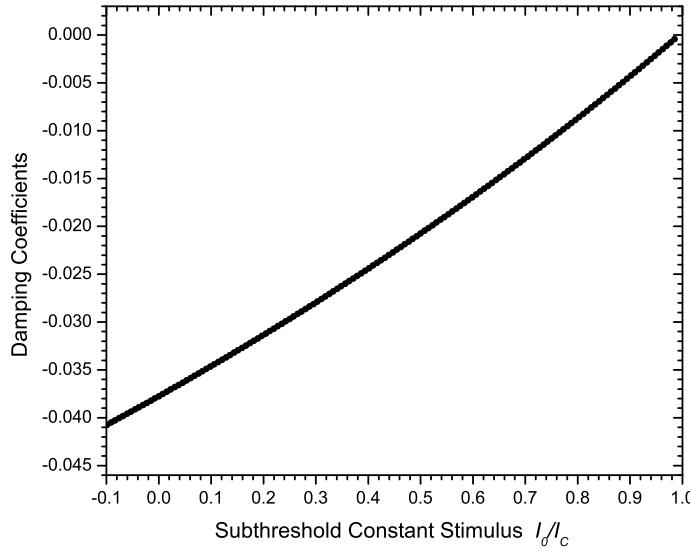


FIG. 1: Theoretical prediction of damping coefficients versus constant stimuli I_0 . I_c is estimated numerically as the critical value of stimulus that make the membrane potential exceed its threshold value $x_{th} = 0$ after a sufficient long time, and chosen as 1.32.

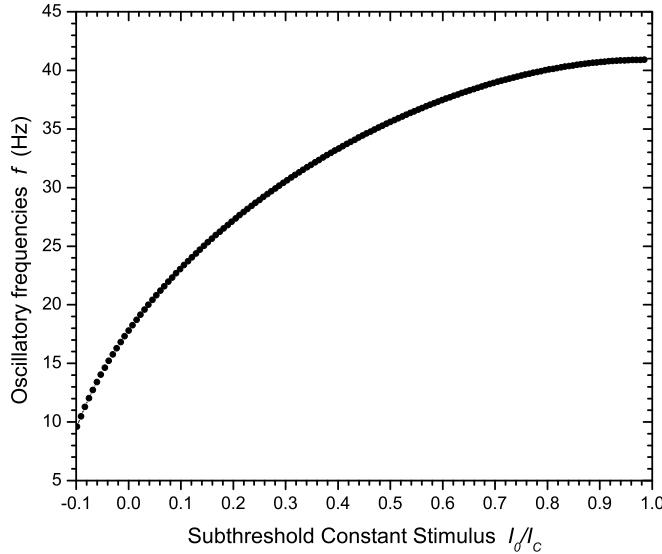


FIG. 2: Theoretical prediction of oscillatory frequencies versus constant stimuli I_0 . I_c is chosen as the same as that in figure 1.